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How does a fundamental string stretch its horizon?

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Abstract: It has recently been shown that if we take into account a class of higher derivative corrections to the effective action of heterotic string theory, the entropy of the black hole solution representing elementary string states correctly reproduces the statistical entropy computed from the degeneracy of elementary string states. So far the form of the solution has been analyzed at distance scales large and small compared to the string scale. We analyze the solution that interpolates between these two limits and point out a subtlety in constructing such a solution due to the presence of higher derivative terms in the effective action. We also study the T-duality transformation rules to relate the moduli fields of the effective field theory to the physical compactification radius in the presence of higher derivative corrections and use these results to find the physical radius of compactification near the horizon of the black hole. The radius approaches a finite value even though the corresponding modulus field vanishes. Finally we discuss the non-leading contribution to the black hole entropy due to space-time quantum corrections to the effective action and the ambiguity involved in comparing this result to the statistical entropy.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

Contents

1. Introduction and summary

The idea that a very massive elementary string state should describe a black hole is quite old [1 − 5]. This leads one to wonder if the entropy associated with these black holes could be given a statistical interpretation as the degeneracy of the elementary string states of a given mass that the black hole represents. One of the problems in carrying out this exercise is that due to large renormalization effects it is often difficult to identify the class of elementary string states that represent a given black hole and vice versa [2, 3].

One way to avoid this problem is to focus attention on BPS states for which the renormalization effects are under control. In particular one can consider heterotic string theory compactified on a torus and consider a fundamental heterotic string wrapped along one of the circles of the torus, carrying w units of winding charge and n units of momentum along the same circle [6, 7]. This describes a BPS state provided we do not excite right-moving world-sheet oscillators. The degeneracy of such states grow as $\exp(4\pi\sqrt{nw})$ for large nw, which suggests that we can assign a statistical entropy of $4\pi\sqrt{nw}$ to these states. On the other hand one can construct extremal BPS black hole solutions carrying the same charge quantum numbers as these states. Thus one might hope that the entropy of the black hole, computed using the Bekenstein-Hawking formula, might reproduce the statistical entropy computed from the degeneracy of the elementary string states. Unfortunately the corresponding black hole has zero area of the event horizon and consequently the Bekenstein-Hawking entropy vanishes [8].

This however is not the end of the story. The black hole solution that gave vanishing entropy was constructed using tree level low energy effective action of the heterotic string theory where we ignore all terms containing more than two derivatives. However if we examine the solution carefully we discover that the Riemann curvature blows up at the horizon and hence the higher derivative terms cannot be ignored. One also finds that in the region where the curvature associated with the string metric is of order unity, the string coupling constant is small for large nw. Thus we expect that for large nw the full solution will receive corrections from higher derivative tree level contribution to the effective action, but the effect of string loop corrections can be ignored.

Although we do not know the precise form of these higher derivative corrections, it was shown in [8] using a simple scaling argument that any correction to the black hole entropy due to these tree level higher derivative terms must be of the form $a\sqrt{nw}$ where a is a purely numerical constant. This clearly agrees with the form of the statistical entropy. However the coefficient a could not be calculated at that time.

Recently in a beautiful paper [9] Dabholkar computed the coefficient a by including in the effective action a class of higher derivative terms. These terms arise from the supersymmetric generalization of the curvature squared term which is known to be present in the tree level effective action of heterotic string theory [10, 11]. Following earlier work [12–17], ref. [9] showed that the black hole solution is modified near the horizon in a way that precisely reproduces the correct value 4π for the coefficient a. In arriving at this result one needs to take into account not only the change in the area of the event horizon (which only accounts for half of the entropy) but also a suitable modification of the Bekenstein-Hawking entropy formula in the presence of the higher derivative terms [18–20].[1] The key assumption behind this construction is that the solution close to the horizon has maximal supersymmetry.

The analysis of [9, 21] gives the form of the solution only very close to the horizon, at distance scale much smaller than the string scale. On the other hand the near horizon solution[2] based on the low energy limit of the effective field theory, described in [8], is expected to be valid only at a distance scale large compared to the string scale where higher derivative terms can be ignored. Thus an important question that arises is: is there a smooth solution that interpolates between these two limits? It turns out that the relevant equation that needs to be analyzed is a second order non-linear differential equation and hence although both the near horizon and the large distance solutions satisfy this equation, it is not obvious that there is a solution that interpolates between the two limiting solutions. One of the goals of this paper will be to analyze this issue. Numerical analysis of the differential equation indicates that if we begin with the near horizon solution and let it evolve according to the equations of motion, the solution does not approach the expected form at large radius, but oscillates about this form. Naively this would indicate

[1] In this context we note that the scaling argument of [8] holds even in the presence of such corrections to the entropy formula. The only assumption required for this argument is that if we change the overall normalization of the action by a constant, then the entropy associated with a given black hole solution gets multiplied by the same constant. This will be reviewed in some detail in section 2.

[2] 'Near horizon' here refers to distance scale small compared to the mass of the black hole.

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that the solution does not approach the desired limit at large radius. However we argue that the supergravity description uses a choice of fields whose propagators have additional poles besides those implied by string theory, and once we make the correct choice of fields by using an appropriate field redefinition, these oscillations disappear and the solution approaches the correct asymptotic form at large radius.[3]

The form of the solution obtained in [9] indicates that the modulus field associated with the radius of the circle along which the fundamental string is wrapped vanishes at the horizon. Naively this would imply that the radius of this circle vanishes at the horizon. However, by analyzing the T-duality transformation laws of various fields we show that the relationship between the physical radius of the circle and the modulus field is modified in such a way that the physical radius approaches a constant at the horizon even though the associated modulus field vanishes.

Although for large charges the string coupling at the horizon is small and hence we can ignore the effect of space-time quantum corrections, ref. [9] analyzes the non-leading contribution to the entropy due to these quantum corrections. We reanalyze these effects and show that if we define the statistical entropy as the logarithm of the degeneracy of states of the elementary string, then the geometric entropy of the black hole fails to reproduce correctly the coefficient of the term proportinal to $\ln(nw)$ in the expression for the statistical entropy. One should however keep in mind that there are alternative definitions of the statistical entropy in terms of other ensembles, e.g. grand canonical ensemble, where we first introduce a grand canonical partition function as a function of the chemical potential conjugate to various charges, and then compute the entropy from this partition function using the usual thermodynamic relations. These two definitions of entropy differ from each other beyond the leading term, and it is not *a priori* clear as to which definition of entropy should be compared to the geometric entropy of the black hole. We show that one such definition of statistical entropy agrees with the geometric entropy of the black hole beyond the leading order approximation.

The paper is organised as follows. We work in the $\alpha' = 16$ unit as in [8, 22]. In section 2 we review the arguments of [8] showing that the black hole entropy has the correct dependence on various parameters up to an overall numerical constant, and also review the recent results of [9, 21]. In section 3 we construct the complete near horizon solution, study the T-duality transformation rules of various fields to determine the relation between the moduli fields and the physical radius of compactification, and discuss the effect of quantum corrections on the black hole entropy. We end in section 4 with some comments on possible generalizations and open issues.

Possible importance of field redefinition in string theory (or equivalently renormalization scheme dependence in two dimensional field theory) in obtaining non-singular solution describing a fundamental string has been discussed earlier in [8, 23]. Modification of black hole solutions and T-duality rules due to higher derivative corrections to the string effective action have been discussed earlier in [24] in a different context.

[3] Although our discussion will focus on the case of two charge black hole representing elementary string states, a similar subtlety is expected to arise for the three charge black hole which has a finite area of the event horizon at the leading order.

2. Supergravity solution for two charge black holes and its near horizon limit

Although the analysis of [9] is able to produce the complete formula for the geometric entropy of the black holes describing elementary string states, it relies on the assumption that the contribution to the geometric entropy comes only from certain higher derivative terms in the effective action. In contrast, the scaling argument of [8] does not rely on any such assumption, and hence is still of interest. In this section we shall first review the scaling argument of ref. [8], and then briefly recall the results of [9].

In [8] we analyzed the most general electrically charged extremal black hole solution in heterotic string theory compactified on T^6. In order to keep our discussion simple, we shall here consider only a special class of black hole solutions representing a heterotic string wound on a circle. For this purpose we take heterotic string theory compactified on $T^5 \times S^1$, T^5 being an arbitrary five-torus and S^1 being a circle of coordinate radius $\sqrt{\alpha'} = 4$. Let us denote by x^μ $(0 \le \mu \le 3)$ the non-compact directions and by x^4 the coordinate along S^1. As in [22] we shall denote by $G^{(10)}_{MN}$, $B^{(10)}_{MN}$ and $\Phi^{(10)}$ the ten dimensional string metric, anti-symmetric tensor field and dilaton respectively. For the description of the black hole solution under study we shall only need to consider non-trivial configurations of the fields $G^{(10)}_{\mu\nu}$, $B^{(10)}_{\mu\nu}$, $G^{(10)}_{4\mu}$, $G^{(10)}_{44}$, $B^{(10)}_{4\mu}$ and $\Phi^{(10)}$. We freeze all other field components to trivial background values, and define:[4]

$$\begin{aligned}
\Phi &= \Phi^{(10)} - \frac{1}{2}\ln(G^{(10)}_{44})\,, \qquad S = e^{-\Phi}\,, \qquad T = \sqrt{G^{(10)}_{44}}\,,\\
G_{\mu\nu} &= G^{(10)}_{\mu\nu} - (G^{(10)}_{44})^{-1}\, G^{(10)}_{4\mu}\, G^{(10)}_{4\nu}\,, \qquad g_{\mu\nu} = e^{-\Phi}\, G_{\mu\nu}\,,\\
A^{(1)}_\mu &= \frac{1}{2}(G^{(10)}_{44})^{-1}\, G^{(10)}_{4\mu}\,, \qquad A^{(2)}_\mu = \frac{1}{2}B^{(10)}_{4\mu}\,,\\
B_{\mu\nu} &= B^{(10)}_{\mu\nu} - 2(A^{(1)}_\mu A^{(2)}_\nu - A^{(1)}_\nu A^{(2)}_\mu)\,.
\end{aligned} \tag{2.1}$$

The low energy effective action involving these fields is then given by [25, 22]

$$\begin{aligned}
\mathcal{S} = \frac{1}{32\pi}\int d^4x\,\sqrt{-\det g}\Big[R &- \frac{1}{2S^2}\, g^{\mu\nu}\,\partial_\mu S\partial_\nu S - \frac{1}{T^2}\, g^{\mu\nu}\,\partial_\mu T\partial_\nu T - \\
&- \frac{1}{12}S^2 g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} H_{\mu\nu\rho}H_{\mu'\nu'\rho'} - ST^2\, g^{\mu\nu}\, g^{\mu'\nu'}\, F^{(1)}_{\mu\mu'}F^{(1)}_{\nu\nu'} - \\
&- ST^{-2}\, g^{\mu\nu}\, g^{\mu'\nu'}\, F^{(2)}_{\mu\mu'}F^{(2)}_{\nu\nu'}\Big]\,,
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
F^{(a)}_{\mu\nu} &= \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu\,, \qquad a = 1, 2\,,\\
H_{\mu\nu\rho} &= \left[\partial_\mu B_{\nu\rho} + 2\left(A^{(1)}_\mu F^{(2)}_{\nu\rho} + A^{(2)}_\mu F^{(1)}_{\nu\rho}\right)\right] + \text{cyclic permutations of } \mu,\ \nu,\ \rho\,.
\end{aligned} \tag{2.3}$$

In this normalization convention the Newton's constant is given by

$$G_N = 2\,. \tag{2.4}$$

[4]Our convention for normalization of the dilaton is the same as that in [8, 22], i.e. e^Φ represents the effective closed string coupling constant.

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Also for $H_{\mu\nu\rho} = 0$ the S- and T-duality transformations take the form [22]:

$$S \to \frac{1}{S}, \qquad F^{(1)}_{\mu\nu} \to -S\, T^{-2} \widetilde{F}^{(2)}_{\mu\nu}, \qquad F^{(2)}_{\mu\nu} \to -S\, T^{2} \widetilde{F}^{(1)}_{\mu\nu}, \tag{2.5}$$

and

$$T \to \frac{1}{T}, \qquad F^{(1)}_{\mu\nu} \to F^{(2)}_{\mu\nu}, \qquad F^{(2)}_{\mu\nu} \to F^{(1)}_{\mu\nu}, \tag{2.6}$$

respectively. $\widetilde{F}^{(a)}_{\mu\nu}$ denotes the Hodge dual of $F^{(a)}_{\mu\nu}$ with respect to the canonical metric $g_{\mu\nu}$.

We now consider an heterotic string wound w times along the circle S^1 labelled by x^4 and carrying n units of momentum along the same circle. Suppose further that asymptotically the four dimensional string coupling takes value g and the radius of S^1 measured in the string metric takes value R. In our normalization convention this imposes the asymptotic conditions:

$$\begin{aligned} & g_{\mu\nu} \to \eta_{\mu\nu} \\ & S \to g^{-2}, \qquad T \to \frac{R}{4}, \\ & F^{(1)}_{\rho t} \to 16\, g^2\, \frac{n}{R^2}\, \frac{1}{\rho^2}, \qquad F^{(2)}_{\rho t} \to \frac{1}{16}\, g^2\, w\, R^2\, \frac{1}{\rho^2}, \end{aligned} \tag{2.7}$$

where ρ is the radial distance from the black hole measured in the canonical metric $g_{\mu\nu}$. An extremal black hole solution satisfying these asymptotic conditions can be read out from the general class of extremal black hole solutions constructed in [26, 8] (see also [27]) and takes the form[5]

$$\begin{aligned} ds_c^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu = -(F(\rho))^{-1/2} \rho dt^2 + (F(\rho))^{1/2}\, \rho^{-1} d\vec{x}^2, \qquad \rho^2 = \vec{x}^2, \\ S &= g^{-2}\, (F(\rho))^{1/2}\, \rho^{-1}, \\ F(\rho) &= \left(\rho + \frac{gwR}{2}\right)(\rho + 8gnR^{-1}), \\ T &= \frac{1}{4}\, R \sqrt{\frac{(\rho + 8gnR^{-1})}{(\rho + gwR/2)}}, \\ F^{(1)}_{\rho t} &= \frac{16 g^2 R^{-2} n}{(\rho + 8gnR^{-1})^2}, \\ F^{(2)}_{\rho t} &= \frac{1}{16}\, \frac{g^2 w R^2}{(\rho + gwR/2)^2}, \\ H_{\mu\nu\rho} &= 0. \end{aligned} \tag{2.8}$$

[5]In using the results of [8] we should note that appropriate components of the right and the left-handed gauge fields given there correspond to $\frac{1}{\sqrt{2}}(A^{(1)}_\mu \pm A^{(2)}_\mu)$ of the present paper, and an appropriate 2×2 block of the matrix M given in [8] can be identified to the matrix $\frac{1}{2}\begin{pmatrix} T^2 + T^{-2} & T^{-2} - T^2 \\ T^{-2} - T^2 & T^2 + T^{-2} \end{pmatrix}$ in the convention of the present paper. In order to produce the solution (2.8) from the one given in [8], we take Q_R, Q_L of [8] to be $2\sqrt{2} g^2 (n/R \pm wR/16)$ and then rescale the fields T, $A^{(1)}_\mu$ and $A^{(2)}_\mu$ by $R/4$, $4/R$ and $R/4$ respectively. The latter operation is a symmetry of the effective action (2.2), and is needed in order to produce a solution for which the asymptotic value of $G^{(10)}_{44}$ is $R^2/16$ so that the asymptotic radius of S^1 is R.

ds_c denote the line element measured in the canonical metric $g_{\mu\nu}$. The line element $ds_{\rm string}$ measured in the string metric $G_{\mu\nu}$ is given by:

$$ds^2_{\rm string} \equiv G_{\mu\nu} dx^\mu dx^\nu = S^{-1} ds_c^2 = -g^2 \rho^2 (F(\rho))^{-1} dt^2 + g^2 d\vec{x}^2 \,. \tag{2.9}$$

The (singular) horizon for this solution is located at $\rho = 0$. The near horizon region is defined as

$$\rho \ll 8gnR^{-1}, \frac{gwR}{2} \,. \tag{2.10}$$

In this region the solution takes the form:

$$\begin{aligned}
ds^2_{\rm string} &= -\frac{\rho^2}{4nw} dt^2 + g^2 d\vec{x}^2 \,, \\
S &= \frac{2\sqrt{nw}}{g\rho} \,, \\
T &= \sqrt{\frac{n}{w}} \,, \\
F^{(1)}_{\rho t} &= \frac{1}{4n} \,, \\
F^{(2)}_{\rho t} &= \frac{1}{4w} \,, \\
ds_c^2 &= -\frac{\rho}{2\, g \sqrt{nw}} dt^2 + \frac{2\, g \sqrt{nw}}{\rho} d\vec{x}^2 \,.
\end{aligned} \tag{2.11}$$

We now introduce rescaled coordinates:

$$\vec{y} = g\, \vec{x} \,, \qquad r = \sqrt{\vec{y}^2} = g\, \rho \,, \qquad \tau = \frac{g^{-1} t}{\sqrt{nw}} \,. \tag{2.12}$$

In this coordinate system the solution near the horizon takes the form:

$$\begin{aligned}
ds^2_{\rm string} &= -\frac{r^2}{4} d\tau^2 + d\vec{y}^2 \,, \qquad r^2 = \vec{y}^2 \,, \\
S &= \frac{2\sqrt{nw}}{r} \,, \\
T &= \sqrt{\frac{n}{w}} \,, \\
F^{(1)}_{r\tau} &= \frac{1}{4} \sqrt{\frac{w}{n}} \,, \\
F^{(2)}_{r\tau} &= \frac{1}{4} \sqrt{\frac{n}{w}} \,.
\end{aligned} \tag{2.13}$$

Notice that in this new coordinate system the solution near the horizon is determined completely by the charge quantum numbers n and w and is independent of the asymptotic value of the moduli g and R. This is an example of the attractor mechanism for supersymmetric black holes [28 – 30].

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We now note that the tree level low energy effective action involving charge neutral fields is invariant under a rescaling of the form:

$$G^{(10)}_{44} \to e^{2\beta} G^{(10)}_{44}\,, \qquad G^{(10)}_{4\mu} \to e^{\beta} G^{(10)}_{4\mu}\,, \qquad B^{(10)}_{4\mu} \to e^{\beta} B^{(10)}_{4\mu}\,, \tag{2.14}$$

keeping the four dimensional dilaton Φ fixed. Physically this corresponds to a rescaling of the compactification radius by e^{β}. Clearly the full string theory is sensitive to the radius of compactification and is not invarinat under this transformation. However the tree level effective action involving charge neutral fields, which are involved in the construction of the black hole solution, is not sensitive to the compactification radius, and the action as well as all the quantities (e.g. the black hole entropy) computed from the effective action will be unchanged under this rescaling. In terms of the four dimensional fields defined in (2.1) this amount to:[6]

$$T \to e^{\beta} T\,, \qquad A^{(1)}_{\mu} \to e^{-\beta} A^{(1)}_{\mu}\,, \qquad A^{(2)}_{\mu} \to e^{\beta} A^{(2)}_{\mu}\,. \tag{2.15}$$

Choosing $e^{\beta} = \sqrt{w/n}$ we can map the near horizon solution (2.13) to:[7]

$$\begin{aligned}
d\check{s}^2_{\text{string}} &= -\frac{r^2}{4}\, d\tau^2 + d\vec{y}^2\,, \qquad r^2 = \vec{y}^2\,, \\
\check{S} &= \frac{2\sqrt{nw}}{r}\,, \\
\check{T} &= 1\,, \\
\check{F}^{(1)}_{r\tau} &= \frac{1}{4}\,, \\
\check{F}^{(2)}_{r\tau} &= \frac{1}{4}\,.
\end{aligned} \tag{2.16}$$

We now note that except for the overall multiplicative factor of $\sqrt{nw}$ in the expression for $\check{S}$, the solution has no dependence on any parameter and is completely universal. We also note that the area of the event horizon, measured in the canonical metric $g_{\mu\nu} = SG_{\mu\nu}$, is given by:

$$A_H = 4\pi r^2 \check{S}|_{r=0} = 8\pi \sqrt{nw}\, r|_{r=0} = 0\,. \tag{2.17}$$

Thus the area of the event horizon vanishes. As a result the black hole entropy also vanishes to this approximation.

Before we proceed we would like to make the following observations:

- (2.16) is an exact solution of the classical low energy supergravity equations of motion. This follows from the fact that (2.8) is a solution of these equations for all n and w, and (2.16) is obtained from this solution by taking the limit $n, w \to \infty$ and carrying out operations which are exact symmetries of the classical low energy supergravity equations of motion.

[6] This is a special case of the O(6,22;R) transformation that was used in [8] to bring the near horizon limit of a general black hole solution into the universal form.

[7] We would like to emphasize that the checked and hatted solutions discussed in this section are related to the original solution (2.11) by transformations which are exact symmetries of the equations of motion of tree level string theory, but are not exact symmetries of the full string theory.

- For $r \gg 1$ the higher derivative corrections to the solution (2.16) are small and we expect the solution of the complete classical equations of motion of string theory to be approximated by (2.16) in this limit. This can be seen by introducing a new coordinate η via the relation $\tau = 2\eta/r$, and writing the solution as

$$\begin{aligned}
\check{ds}^2_{\text{string}} &= -d\eta^2 + d\vec{y}^2 + 2\,\frac{\eta}{r}d\eta dr - \frac{\eta^2}{r^2}dr^2\,, \qquad r^2 = \vec{y}^2\,,\\
\frac{\partial_r \check{S}}{\check{S}} &= -\frac{1}{r}\,,\\
\check{T} &= 1\,,\\
\check{F}^{(1)}_{r\eta} &= \frac{1}{2r}\,,\\
\check{F}^{(2)}_{r\eta} &= \frac{1}{2r}\,.
\end{aligned} \tag{2.18}$$

Thus we see that for fixed η, the metric approaches flat metric and all other fields become trivial for large r. Thus we expect the corrections due to higher derivative terms to be small. In fact from the structure of (2.18) it is clear that for fixed η each derivative with respect to r brings down a factor of $1/r$ and hence the effect of the four derivative terms in the action is suppressed by a factor of $1/r^2$ relative to the two derivative terms. Thus we expect that the modification of the solution (2.16) due to the higher derivative terms will be of order $1/r^2$ relative to the leading term. This observation will be useful for our analysis later.

Let us now consider the effect of various corrections to the effective action [8]. First of all we see that $S \to \infty$ as $r \to 0$ and even for $r \sim 1$, S is of order $\sqrt{nw}$ which is large for large-N and w. Since S measures the inverse of the string coupling we conclude that stringy quantum corrections can be ignored for large-N and w [8]. On the other hand since various curvatures are of order unity for $r \sim 1$ we expect that the tree level higher derivative terms will affect the solution and the entropy. To study the general form of these corrections, we recall that the complete tree level effective action of the heterotic string theory in the subsector under study has the form:

$$\mathcal{S} = \int d^4x\,\sqrt{-\det G}\,S\,\mathcal{L}\Big(G_{\mu\nu}, B_{\mu\nu}, T, A^{(1)}_\mu, A^{(2)}_\mu, \frac{\partial_\mu S}{S}\Big)\,. \tag{2.19}$$

Note in particular that under multiplication of S by a constant, the action gets multiplied by the same constant. This shows that given any solution of the full equations of motion derived from the action (2.19), we can get another solution by multiplying S by an arbitrary constant, leaving the rest of the fields unchanged. Thus in order to study possible corrections to the solution (2.17) due to the higher derivative terms in the action (2.19), we could first find corrections to a different solution

$$\begin{aligned}
\hat{ds}^2_{\text{string}} &= -\frac{r^2}{4}\,d\tau^2 + d\vec{y}^2\,, \qquad r^2 = \vec{y}^2\,,\\
\hat{S} &= \frac{2}{r}\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{T} &= 1\,, \\
\hat{F}^{(1)}_{r\tau} &= \frac{1}{4}\,, \\
\hat{F}^{(2)}_{r\tau} &= \frac{1}{4}\,,
\end{aligned}
\tag{2.20}
$$

and then multiply the $\hat{S}$ for the resulting solution by $\sqrt{nw}$ to find the correction to (2.16). Since (2.20) has a completely universal form without any parameter. and since furthermore the action (2.19) is also completely universal, it is clear that the higher derivative terms in (2.19) will change (2.20) to a universal form:

$$
\begin{aligned}
\hat{ds}^2_{\text{string}} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,d\vec{y}^2\,, \qquad r^2 = \vec{y}^2\,, \\
\hat{S} &= f_3(r)\,, \\
\hat{T} &= f_4(r)\,, \\
\hat{F}^{(1)}_{r\tau} &= f_5(r)\,, \\
\hat{F}^{(2)}_{r\tau} &= f_6(r)\,,
\end{aligned}
\tag{2.21}
$$

where $f_1(r), \ldots, f_6(r)$ are a set of universal functions. This particular parametrization has been chosen for later convenience. For large r these functions must agree with the solution (2.20). This gives

$$
f_1(r) \simeq \frac{r}{2}\,, \qquad f_2(r) \simeq \frac{2}{r}\,, \qquad f_3(r) \simeq \frac{2}{r}\,, \qquad f_4(r) \simeq 1\,, \qquad f_5(r) \simeq \frac{1}{4}\,, \qquad f_6(r) \simeq \frac{1}{4}\,.
\tag{2.22}
$$

The higher derivative corrections to (2.16) is now generated by multiplying S in (2.21) by a factor of $\sqrt{nw}$:

$$
\begin{aligned}
\check{ds}^2_{\text{string}} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,d\vec{y}^2\,, \qquad r^2 = \vec{y}^2\,, \\
\check{S} &= \sqrt{nw}\,f_3(r)\,, \\
\check{T} &= f_4(r)\,, \\
\check{F}^{(1)}_{r\tau} &= f_5(r)\,, \\
\check{F}^{(2)}_{r\tau} &= f_6(r)\,.
\end{aligned}
\tag{2.23}
$$

Using the inverse of the transformation (2.15) we can now generate the modified version of the solution (2.13):

$$
\begin{aligned}
ds^2_{\text{string}} &= -\frac{f_1(r)}{f_3(r)}\,d\tau^2 + \frac{f_2(r)}{f_3(r)}\,d\vec{y}^2\,, \qquad r^2 = \vec{y}^2\,, \\
S &= \sqrt{nw}\,f_3(r)\,, \\
T &= \sqrt{\frac{n}{w}}\,f_4(r)\,, \\
F^{(1)}_{r\tau} &= \sqrt{\frac{w}{n}}\,f_5(r)\,, \\
F^{(2)}_{r\tau} &= \sqrt{\frac{n}{w}}\,f_6(r)\,.
\end{aligned}
\tag{2.24}
$$

We now turn to the computation of entropy associated with this solution. In the presence of higher derivative corrections the entropy is no longer proportional to the area of the event horizon; there are additional corrections [18 − 20]. These corrections all have the property that if the action is multiplied by a constant then the entropy associated with a given solution also gets multiplied by the same constant. Now suppose a denote the entropy associated with the solution (2.21). Then since the solution (2.21) and the action (2.19) are both universal, a must be a purely numerical coefficient. Since (2.23) differs from (2.21) in a multiplicative factor of $\sqrt{nw}$ in the expression for S, and since from (2.19) we see that the effect of this multiplicative factor is to multiply the action by $\sqrt{nw}$, the entropy associated with the solution (2.23) must be given by [8]:

$$S_{BH} = a\sqrt{nw}\,. \tag{2.25}$$

Since (2.23) and (2.24) are related by the transformation (2.15) which is an exact symmetry of the tree level effective action, (2.25) also gives the entropy associated with the solution (2.24).

On the other hand counting of states of fundamental heterotic string carrying w units of winding and n units of momentum along S^1 shows that for large-N and w the degeneracy of states grows as $e^{4\pi\sqrt{nw}}$. Thus the statistical entropy, defined as the logarithm of the degeneracy of states, is given by:

$$S_{\rm stat} \simeq 4\pi\sqrt{nw}\,, \tag{2.26}$$

for large-N and w. Thus we see that up to an overall multiplicative constant the statistical entropy agrees with the Bekenstein-Hawking entropy of the black hole [8].

For later use, it will be convenient to use (2.12) to rewrite (2.24) in terms of the original variables ρ and t:

$$\begin{aligned}
ds^2_{\rm string} &= -\frac{1}{g^2 nw}\,\frac{f_1(g\rho)}{f_3(g\rho)}dt^2 + g^2\,\frac{f_2(g\rho)}{f_3(g\rho)}\,d\vec{x}^2\,, \qquad \rho=\sqrt{\vec{x}^2}\\
S &= \sqrt{nw}f_3(g\rho)\,,\\
T &= \sqrt{\frac{n}{w}}\,f_4(g\rho)\,,\\
F^{(1)}_{\rho t} &= \frac{1}{n}\,f_5(g\rho)\,,\\
F^{(2)}_{\rho t} &= \frac{1}{w}\,f_6(g\rho)\,,\\
ds^2_c &= S\,ds^2_{\rm string} = -\frac{1}{g^2\sqrt{nw}}\,f_1(g\rho)\,dt^2 + g^2\,\sqrt{nw}\,f_2(g\rho)\,d\vec{x}^2\,.
\end{aligned} \tag{2.27}$$

This finishes our review of [8]. Let us now briefly mention the recent results of refs. [9, 21]. In these papers the authors compute the value of the coefficient a by taking into account a special class of higher derivative terms in the effective action which are required for the supersymmetric completion of the curvature squared term [10] that is known to be present in the tree level effective action of the heterotic string theory [11]. Based on earlier work [13 − 17, 31, 32] these papers concluded that in the presence of the higher derivative terms the solution near $r = 0$ gets modified in such a way that the horizon acquires a finite area. The naive entropy computed from this using the Bekenstein-Hawking formula is $2\pi\sqrt{nw}$. However, as was shown in [9, 21], there are corrections to the entropy formula due to the presence of the higher derivative terms in the action, and these give an additonal contribution of $2\pi\sqrt{nw}$. Thus the net entropy of the extremal black hole is given by $4\pi\sqrt{nw}$ in agreement with the statistical entropy (2.26).

The analysis of [9, 21] was based on the assumption that at the horizon the black hole solution develops enhanced supersymmetry. While this leads to the solution close to the horizon, this does not give us any information about the interpolating functions $f_1(r)$, ... $f_6(r)$ for finite values of r. In the next section we shall study the complete solution in the presence of this special class of higher derivative terms, and find the functions $f_1(r), \ldots f_6(r)$ which interpolate between the large r limit discussed in this section and the small r results of refs. [9, 21].

3. Modification of the solution by higher derivative terms and its near horizon limit

In this section we shall find the modification of the solution (2.8) by taking into account a special class of higher derivative corrections to the effective action. In order to do so, we need to first rewrite the low energy effective action (2.2) in the language of $N = 2$ supergravity and then analyze the effect of higher derivative corrections.

3.1 The low energy effective action as $N = 2$ supergravity

The action of $N = 2$ supergravity coupled to n vector multiplets is governed by a prepotential F which is a function of $(n + 1)$ complex scalars X^I $(0 \le I \le n)$. The X^I's are projective coordinates and F is a homogeneous function of the X^I's of degree 2. The gauge invariant bosonic degrees of freedom are the metric $g_{\mu\nu}$, the complex scalars X^I/X^0, and a set of $(n+1)$ gauge fields $\mathcal{A}^I_\mu$. Let us define

$$\mathcal{F}^I_{\mu\nu} \equiv \partial_\mu \mathcal{A}^I_\nu - \partial_\nu \mathcal{A}^I_\mu \,, \tag{3.1}$$

$$F_I \equiv \frac{\partial F}{\partial X^I}\,, \qquad F_{IJ} \equiv \frac{\partial^2 F}{\partial X^I \partial X^J}\,, \tag{3.2}$$

$$L_I \equiv (F_{IJ}\bar{X}^J - \bar{F}_I)\,, \qquad L \equiv \bar{F} - \frac{1}{2} F_{IJ}\, \bar{X}^I \bar{X}^J\,, \qquad \mathcal{N}_{IJ} \equiv \frac{1}{4} i \left(F_{IJ} + \frac{1}{2}\frac{L_I L_J}{L} \right), \tag{3.3}$$

$$e^{-\mathcal{K}} \equiv i(\bar{X}^I F_I - X^I \bar{F}_I)\,, \tag{3.4}$$

$$\mathcal{F}^{I\pm}_{\mu\nu} \equiv \frac{1}{2}\left(\mathcal{F}^I_{\mu\nu} \pm i \widetilde{\mathcal{F}}^I_{\mu\nu} \right). \tag{3.5}$$

$$\mathcal{G}^{I\pm}_{\mu\nu} \equiv \pm 16\pi\, i \left(\sqrt{-\det g} \right)^{-1} \frac{\delta S}{\delta \mathcal{F}^{I\pm}_{\mu\nu}}\,, \qquad \mathcal{G}^I_{\mu\nu} \equiv \mathcal{G}^{I+}_{\mu\nu} + \mathcal{G}^{I-}_{\mu\nu}\,, \tag{3.6}$$

where $\widetilde{\mathcal{F}}^I_{\mu\nu}$ denotes the Hodge dual of $\mathcal{F}^I_{\mu\nu}$, and in computing $\delta S/\delta \mathcal{F}^{I\pm}_{\mu\nu}$ we need to treat $\mathcal{F}^{I\pm}_{\mu\nu}$ as independent variables. Since X^I's are projective coordinates, we can impose a gauge condition on the X^I's. The convenient gauge choice is the $e^{-\mathcal{K}}$ = constant gauge.

In this gauge the bosonic part of the action takes the form [33, 13, 17]:[8]

$$\mathcal{S} = \frac{1}{8\pi}\int d^4x\sqrt{-\det g}\biggl[\frac{1}{2}\,e^{-\mathcal{K}}R - ig^{\mu\nu}(\partial_\mu X^I\partial_\nu\bar{F}_I - \partial_\mu\bar{X}^I\partial_\nu F_I) + \Bigl\{\mathcal{N}_{IJ}g^{\mu\mu'}g^{\nu\nu'}\mathcal{F}^{I-}_{\mu\nu}\mathcal{F}^{J-}_{\mu'\nu'} + h.c.\Bigr\}\biggr], \tag{3.7}$$

For the system we are considering, the prepotential is [35]

$$F = -\frac{X^1(X^2)^2}{X^0}\,. \tag{3.8}$$

If we define the gauge invariant fields S and T through

$$\frac{X^1}{X^0} = i\,S\,, \qquad \frac{X^2}{X^0} = i\,T\,, \tag{3.9}$$

and choose the gauge condition

$$e^{-\mathcal{K}} = \frac{1}{2}\,, \tag{3.10}$$

then from eqs. (3.4), (3.8) we get

$$X^0 = \frac{1}{4\,T\,\sqrt{S}}\,. \tag{3.11}$$

For real S and T eqs. (3.1)–(3.7) now produce the action:

$$\begin{aligned}\mathcal{S} = \frac{1}{32\pi}\int d^4x\sqrt{-\det g}\biggl[&R - \frac{1}{2S^2}\,g^{\mu\nu}\,\partial_\mu S\partial_\nu S - \frac{1}{T^2}\,g^{\mu\nu}\,\partial_\mu T\partial_\nu T - \\ &- ST^2\,g^{\mu\nu}\,g^{\mu'\nu'}\,\mathcal{F}^0_{\mu\mu'}\mathcal{F}^0_{\nu\nu'} - S^{-1}T^2\,g^{\mu\nu}\,g^{\mu'\nu'}\,\mathcal{F}^1_{\mu\mu'}\mathcal{F}^1_{\nu\nu'} - \\ &- 2\,S\,g^{\mu\nu}\,g^{\mu'\nu'}\,\mathcal{F}^2_{\mu\mu'}\,\mathcal{F}^2_{\nu\nu'}\biggr]\,.\end{aligned} \tag{3.12}$$

This agrees with the action (2.2) after a duality transformation on the field $\mathcal{F}^1_{\mu\nu}$ if we make the identification:[9]

$$F^{(1)}_{\mu\nu} = \mathcal{F}^0_{\mu\nu}\,, \qquad F^{(2)}_{\mu\nu} = \mathcal{G}^1_{\mu\nu} = \frac{T^2}{S}\,\widetilde{\mathcal{F}}^1_{\mu\nu}\,, \tag{3.13}$$

and set $\mathcal{A}^2_\mu$ to 0. For a general configuration of X^I's and $\mathcal{A}^I_\mu$'s the imaginary parts of T and S can be identified respectively with an appropriate off-diagonal component of the internal metric and the axion field obtained by dualizing the field $B_{\mu\nu}$, whereas $\mathcal{A}^2_\mu$ can be regarded as an appropriate linear combination of $G^{(10)}_{m\mu}$ and $B^{(10)}_{m\mu}$ for $5 \le m \le 9$. During the rest of our analysis we shall consider configurations where S and T are real.

[8] In writing down this action we have implicitly assumed a reality condition on the fields such that the fields X^I are either purely real or purely imaginary and the prepotential F is purely imaginary. In the present example this amounts to restricting the fields S and T defined in (3.9) to be real. Otherwise there will be additional contribution to the kinetic term for the scalar fields. I would like to thank S. Das for drawing my attention to this issue.

[9] Note that heterotic string theory compactified on T^6 has $N = 4$ supersymmetry, but here we are considering a truncated version of the theory which has $N = 2$ supersymmetry.

3.2 Higher derivative corrections

The corrections associated with supersymmetrization of the curvature squared term can be taken into account by modifying the prepotential to [13, 17]

$$F = -\frac{X^1(X^2)^2}{X^0} + \widehat{A}\, f\left(\frac{X^1}{X^0}\right), \tag{3.14}$$

where $\widehat{A}$ is a background chiral superfield whose highest component contains the square of the Weyl tensor, and f is a function to be specified later (see eqs. (3.49) and (3.72) below). It has the property

$$f(iS) + (f(iS))^* = 0 \quad \text{for real } S\,. \tag{3.15}$$

We define

$$F_{\widehat{A}} = \frac{\partial F}{\partial \widehat{A}} = f\left(\frac{X^1}{X^0}\right), \tag{3.16}$$

and various other quantities as in eqs. (3.1)–(3.6). An expression for the bosonic part of the action for a general prepotential $F(X^I, \widehat{A})$ has been given in [17], but we shall not review it here. Instead we shall focus our attention on a class of $N = 1$ supersymmetric black hole solutions constructed in [17, 34] after taking into account the corrections given in (3.16). In an appropriate gauge these solutions have the form:[10]

$$ds_c^2 = -e^{2G(\rho)}dt^2 + e^{-2G(\rho)}d\vec{x}^2\,, \qquad \rho = \sqrt{\vec{x}^2}\,, \tag{3.17}$$

$$e^{-G}(X^I - \bar{X}^I) = i\left(a_I + \frac{b_I}{\rho}\right), \tag{3.18}$$

$$e^{-G}(F_I - \bar{F}_I) = i\left(c_I + \frac{d_I}{\rho}\right), \tag{3.19}$$

where a_I, b_I, c_I, d_I are arbitrary real constants,[11]

$$\mathcal{F}^I_{\rho t} = \partial_\rho(e^G(X^I + \bar{X}^I))\,, \tag{3.20}$$

$$\mathcal{G}^I_{\rho t} = \partial_\rho(e^G(F_I + \bar{F}_I))\,, \tag{3.21}$$

$$\widehat{A} = -64\, e^{2G}\,(\partial_\rho G)^2\,, \tag{3.22}$$

$$e^{-\mathcal{K}} + \frac{1}{2}\chi = -128 i e^{3G}\frac{1}{\rho^2}\partial_\rho\left(\rho^2 e^{-G}\partial_\rho G\,(F_{\widehat{A}} - \bar{F}_{\widehat{A}})\right). \tag{3.23}$$

Here χ is an arbitrary constant whose value is determined by the gauge condition. We shall choose the gauge

$$\chi = -1\,, \tag{3.24}$$

so that in the absence of coupling to the background superfield $\widehat{A}$ the gauge condition agrees with (3.10).

[10] Ref. [17] considered a more general class of solutions by allowing the supersymmetry transformation parameter to rotate by a phase as we move in space. Since we shall be interested in a solution for which the fields S and T are real we shall set the phase to 1.

[11] Physically the constants a_I and c_I measure the asymptotic values of various fields whereas b_I and d_I measure the charges carried by the black hole.

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The procedure for solving these equations is as follows. For given constants a_I, b_I, c_I, d_I, eqs. (3.18), (3.19) give $2n$ real equations which can be used to used to solve for the n complex X^I's in terms of G and $\widehat{A}$. (3.22) gives $\widehat{A}$ in terms of G. Substituting these in (3.23) we get a differential equation for G which can then be solved. Once G and the X^I's have been found, we can use (3.20), (3.21) to calculate the gauge field strengths $\mathcal{F}^I_{\rho t}$ and $\mathcal{G}^I_{\rho t}$.

Thus in order to find the black hole solution describing the elementary string states, we first need to determine the constants a_I, b_I, c_I and d_I. For this we note that in the absence of the coupling to the background superfield $\widehat{A}$, i.e. for $f(X^1/X^0) = 0$, the solution (2.8) has the form given in eqs. (3.17)–(3.24) for the following choice of the constants:

$$\begin{aligned} &a_1 = 2g^{-1}R^{-1}\,, \qquad b_1 = w\,, \qquad c_0 = -\frac{R}{8}g^{-1}\,, \qquad d_0 = -n\,, \qquad a_2 = \frac{1}{2}\,g\,, \\ &a_0 = b_0 = b_2 = 0\,, \qquad c_I = d_I = 0 \qquad \text{for } I = 1, 2\,. \end{aligned} \tag{3.25}$$

In order to study the modification of the solution due to coupling to the background superfield $\widehat{A}$, we first note that for a given solution the constants a_I, b_I, c_I, d_I may be determined by knowing the form of the solution at large ρ to order $1/\rho$. As argued in the last section, the modification of the solution due to the four and higher derivative terms in the action appear at order $1/\rho^2$. Hence the constants a_I, b_I, c_I and d_I should not change due to the higher derivative corrections and must have the same values as given in eq. (3.25). With this choice X^0 is real and X^1, X^2 are purely imaginary, and the non-trivial components of eqs. (3.17)–(3.22) may be expressed as:[12]

$$\begin{aligned} e^{-G}\,X^0\,S &= \frac{1}{2}\left(2g^{-1}R^{-1} + \frac{w}{\rho}\right) \\ e^{-G}\,X^0\,T &= \frac{g}{4}\,, \\ e^{-G}\,X^0\,S\left(T^2 + \frac{\widehat{A}}{(X^0)^2}\,f'(iS)\right) &= \frac{1}{2}\left(\frac{R}{8}g^{-1} + \frac{n}{\rho}\right), \end{aligned} \tag{3.26}$$

$$\widehat{A} = -64\,e^{2G}\,(\partial_\rho G)^2\,, \tag{3.27}$$

$$\begin{aligned} F^{(1)}_{\rho t} &= \mathcal{F}^0_{\rho t} = 2\partial_\rho\left(e^G X^0\right), \\ F^{(2)}_{\rho t} &= \mathcal{G}^1_{\rho t} = 2\partial_\rho\left[e^G X^0\left(T^2 + \frac{\widehat{A}}{(X^0)^2}\,f'(iS)\right)\right]. \end{aligned} \tag{3.28}$$

For the choice of prepotential given in (3.14), we have, using (3.15),

$$e^{-\mathcal{K}} \equiv i(\bar{X}^I F_I - X^I \bar{F}_I) = 4\,S\left(2\,T^2 + \frac{\widehat{A}}{(X^0)^2}f'(iS)\right)(X^0)^2\,. \tag{3.29}$$

Eqs. (3.23), (3.24) now give:

$$4\,S\left(2\,T^2 + \frac{\widehat{A}}{(X^0)^2}f'(iS)\right)(X^0)^2 = \frac{1}{2} - 256\,i\,e^{3G}\,\frac{1}{\rho^2}\,\partial_\rho\left[\rho^2\,e^{-G}\,\partial_\rho G\,f(iS)\right]. \tag{3.30}$$

[12] We continue to identify S, T, $F^{(1)}_{\mu\nu}$ and $F^{(2)}_{\mu\nu}$ with $-iX^1/X^0$, $-iX^2/X^0$, $\mathcal{F}^0_{\mu\nu}$ and $\mathcal{G}^1_{\mu\nu}$ respectively, but the fields defined this way may no longer be related to the ten dimensional fields via eqs. (2.1), (2.3). We shall elaborate on this in section 3.4.

In order to solve these equations, we can first use eqs. (3.26), (3.27) to express S, T, X^0 and $\widehat{A}$ in terms of G, and then substitute these into (3.30) to get a second order non-linear differential equation for G.

We shall now study various aspects of these equations.

3.3 Near horizon geometry and entropy

In the $\rho \to 0$ limit we can rewrite the first two equations in (3.26) as:

$$\begin{aligned} X^0 &\simeq \frac{w}{2\rho}\, e^G\, S^{-1}\,, \\ T &\simeq \frac{g}{2w}\, \rho\, S\,. \end{aligned} \tag{3.31}$$

Using (3.27) and (3.31) the last equation of (3.26) now gives

$$\frac{\rho^2}{w^2}\, S^2 \left[\frac{g^2}{4} - 256\, (\partial_\rho G)^2\, f'(iS)\right] \simeq \frac{n}{w}\,. \tag{3.32}$$

On the other hand eq. (3.30) takes the form:

$$S\, e^{2G} \left[\frac{g^2}{2} - 256\, (\partial_\rho G)^2 f'(iS)\right] = \frac{1}{2} - 256\, i\, e^{3G}\, \frac{1}{\rho^2} \partial_\rho \left[\rho^2\, e^{-G}\, \partial_\rho G\, f(iS)\right]. \tag{3.33}$$

If we take the following ansatz for the solutions near $\rho = 0$

$$S \simeq S_0\,, \qquad e^{2G} \simeq K_0 \rho^2\,, \tag{3.34}$$

then by substituting this into (3.32), (3.33) we get

$$S_0^2\, f'(iS_0) = -\frac{1}{256}\, n w\,, \qquad S_0\, K_0\, f'(iS_0) = -\frac{1}{512}\,. \tag{3.35}$$

For a given function f these equations can be solved to find S_0 and K_0. Using (3.31), (3.27), (3.28) and (3.35) we get

$$X^0 \simeq \frac{w}{2}\, K_0^{1/2}\, S_0^{-1}\,, \qquad T \simeq \frac{g}{2w}\, S_0\, \rho\,, \qquad F^{(1)}_{\rho t} \simeq \frac{1}{2n}\,, \qquad F^{(2)}_{\rho t} \simeq \frac{1}{2w}\,. \tag{3.36}$$

This determines the field configuration near $\rho = 0$.

Substituting (3.34) into the expression for the metric given in (3.17) we see that the area of the event horizon, measured in the canonical metric, is

$$A_H = 4\, \pi\, K_0^{-1}\,. \tag{3.37}$$

Thus the naive black hole entropy will be given by

$$\frac{A_H}{4 G_N} = \frac{\pi}{2} K_0^{-1}\,, \tag{3.38}$$

where we have used $G_N = 2$ as given in (2.4). However as shown in [13], due to the presence of higher derivative terms in the action this expression gets modified to

$$S_{BH} = \frac{A_H}{4 G_N} - 256\, \pi\, \mathrm{Im}(F_{\widehat{A}})\,, \tag{3.39}$$

where $F_{\widehat{A}}$ has been defined in (3.16). Using the expression (3.14) for F, and eqs. (3.37), (2.4), (3.35) we can express (3.39) as:

$$S_{BH} = \frac{1}{2}\pi K_0^{-1} - 256\pi\ \mathrm{Im}(f(iS_0)) = -256\pi\left(S_0 f'(iS_0) + \mathrm{Im}(f(iS_0))\right). \tag{3.40}$$

Thus once K_0 and S_0 have been determined from (3.35), eq. (3.40) can be used to compute the black hole entropy. Note that although eqs. (3.26)–(3.30) represent the condition for preserving half of the space-time supersymmetries of the vacuum, the solution (3.35), (3.36) at the horizon $\rho = 0$ actually preserves larger number of supersymmetries [13, 17].

There is however a subtle point that we have overlooked. The remark below (3.30) shows that G satisfies a second order non-linear differential equation. (3.34), (3.35) describes a particular solution of this equation near $\rho = 0$. In order to show that this describes the correct behaviour of the *black hole solution* near $\rho = 0$, we need to ensure that this solution approaches the correct asymptotic form (2.8) for large ρ where the effect of higher derivative corrections should be negligible. Since a general solution of the differential equation has two integration constants, there is no *a priori* guarantee that the choice of integration constants which lead to the form given in (3.34), (3.35) will also have the correct asymptotic behaviour. We shall return to this issue in section 3.6.

3.4 T-duality

The near horizon expression for T given in (3.36) shows that it vanishes as $\rho \to 0$. If $G^{(10)}_{44}$ is identified with T^2 as in eq. (2.1), then this would imply that $G^{(10)}_{44}$ would vanish as $\rho \to 0$. This is somewhat surprising if we consider the fact that before including the higher derivative corrections to the action, $G^{(10)}_{44}$ approached a finite value $T^2 = \frac{n}{w}$ as $\rho \to 0$ (see eq. (2.11)). Since unlike the field S, $G^{(10)}_{44}$ had already reached a fixed point value, one would have expected that higher derivative corrections would not drastically modify this behaviour.

We shall now argue that after inclusion of the higher derivative corrections given in (3.14), the correct identification of $G^{(10)}_{44}$ is not T^2, but,

$$G^{(10)}_{44} = T^2 + \frac{\widehat{A}}{(X^0)^2} f'(iS). \tag{3.41}$$

In that case the first and the last equations in (3.26) show that in the $\rho \to 0$ limit

$$G^{(10)}_{44} \to \frac{n}{w}. \tag{3.42}$$

This agrees with the near horizon value of $G^{(10)}_{44}$ in the absence of higher derivative corrections.

In order to establish (3.41) we need to study how the T-duality transformation

$$G^{(10)}_{44} \to (G^{(10)}_{44})^{-1}, \tag{3.43}$$

which is an exact symmetry of heterotic string theory on $T^5 \times S^1$, is realized in terms of the fields X^I. According to [35] this corresponds to the transformation:

$$
\begin{aligned}
X^0 &\to \widetilde{X}^0 = -F_1 = \frac{(X^2)^2}{X^0} - \frac{\widehat{A}}{X^0}\, f'(iS)\,, \\
X^1 &\to \widetilde{X}^1 = F_0 = \frac{X^1\,(X^2)^2}{(X^0)^2} - \frac{\widehat{A}\,X^1}{(X^0)^2}\, f'(iS)\,, \\
X^2 &\to \widetilde{X}^2 = -X^2\,,
\end{aligned} \tag{3.44}
$$

$$
\begin{aligned}
F_0 &\to \widetilde{F}_0 = X^1\,, \\
F_1 &\to \widetilde{F}_1 = -X^0\,, \\
F_2 &\to \widetilde{F}_2 = -F_2 = -\frac{2X^1X^2}{X^0}\,,
\end{aligned} \tag{3.45}
$$

where

$$
\widetilde{F}_I \equiv F_I(\{\widetilde{X}^I\}, \widehat{A})\,. \tag{3.46}
$$

Eqs. (3.44) describes the transformation laws of the fields X^I, whereas eqs. (3.45) are consistency conditions which must be satisfied in order that the transformations (3.44) are symmetries of the equations of motion. It can be easily verified that eqs. (3.45) follow from eqs. (3.44).

Using (3.44) we see that

$$
\begin{aligned}
\widetilde{S} &= -i\,\frac{\widetilde{X}^1}{\widetilde{X}^0} = -i\,\frac{X^1}{X^0} = S\,, \\
\widetilde{T} &= -i\frac{\widetilde{X}^2}{\widetilde{X}^0} = i\frac{X^2}{\frac{(X^2)^2}{X^0} - \frac{\widehat{A}}{X^0}\, f'(iS)} = \frac{T}{T^2 + \frac{\widehat{A}}{(X^0)^2}\, f'(iS)}\,.
\end{aligned} \tag{3.47}
$$

In the absence of higher derivative corrections, i.e. when $f(iS) = 0$, this gives the familiar $T \to T^{-1}$ duality transformation. However we see that the duality transformation law of T gets modified by the higher derivative terms. Thus T can no longer by identified as $\sqrt{G^{(10)}_{44}}$ which transforms as (3.43) even when higher derivative corrections are included. On the other hand we note from (3.44) that

$$
\widetilde{T}^2 + \frac{\widehat{A}}{(\widetilde{X}^0)^2}\, f'(i\widetilde{S}) = \frac{1}{T^2 + \frac{\widehat{A}}{(X^0)^2}\, f'(iS)}\,. \tag{3.48}
$$

Comparing this to (3.43), and by using the requirement that for $f(iS) = 0$, $G^{(10)}_{44}$ should reduce to T^2, we reach the identification given in (3.41).

3.5 Tree level heterotic string theory and universality

The higher derivative corrections to the tree level effective action of heterotic string theory are given by the following choice of the function $f(u)$ [9]:

$$
f(u) = -\frac{C}{64}\, u\,, \qquad C = 1\,. \tag{3.49}
$$

Although the constant C is equal to unity, we shall analyze the solution assuming that it is an arbitrary constant so that at various stages we can recover the leading α' result by setting C to 0. Eq. (3.35) now gives:

$$S_0 = \frac{1}{2}\sqrt{\frac{nw}{C}}\,, \qquad K_0 = \frac{1}{4\sqrt{C\,nw}}\,. \tag{3.50}$$

Using eqs. (3.40) and (3.50) we get:

$$S_{BH} = 4\,\pi\,\sqrt{C\,nw}\,. \tag{3.51}$$

For $C = 1$ this reproduces the microscoic entropy (2.26) in agreement with the results of [9]. Setting $C = 0$ we recover the supergravity result that the entropy vanishes.

We shall now explicitly check that this solution reproduces the scaling property encoded in eq. (2.27) in the limit of large-N and w. For this we take the limit of large-N and w in eqs. (3.26)–(3.30), substitute the general form (2.27) into these equations, and use the form (3.49) for $f(u)$. First of all comparison between the form of the metric (2.27) and (3.17) gives

$$f_1(r) f_2(r) = 1\,. \tag{3.52}$$

Eqs. (3.26)–(3.30) in this limit give

$$\begin{aligned}
X^0 &= \frac{1}{2g\rho}\sqrt{\frac{w}{n}}\,(nw)^{-1/4}\,\frac{\sqrt{f_1(g\rho)}}{f_3(g\rho)}\,, \qquad (3.53)\\
\frac{f_3(r)}{f_4(r)} &= \frac{2}{r}\,,\\
f_3(r)\,f_4(r)\left[1+4\,C\left(\frac{f_1'(r)}{f_1(r)}\right)^2\right] &= \frac{2}{r}\,,\\
\frac{1}{2}\,f_1(r)\,f_3(r) &= \frac{1}{2} - \frac{2\,C}{r^2} f_1(r)\,\partial_r\left[r^2\,f_3(r)\,\frac{f_1'(r)}{f_1(r)}\right],\\
f_5(r) &= \partial_r\left(\frac{1}{r}\,\frac{f_1(r)}{f_3(r)}\right),\\
f_6(r) &= \partial_r\left(\frac{1}{r}\,\frac{f_1(r)}{f_3(r)}\right). \qquad (3.54)
\end{aligned}$$

We see that eqs. (3.52) and (3.54) involving the functions $f_i(r)$ are completely independent of any external parameters, as predicted by the scaling argument.

We can try to solve these equations by introducing a new function $h(r)$ through

$$f_1(r) = e^{h(r)}\,. \tag{3.55}$$

Then (3.52) and the first two and the last two equations of (3.54) give[13]

$$f_2(r) = e^{-h(r)}\,,$$

[13]We could also have gotten eqs. (3.55)–(3.57) by directly substituting the general form (2.21) into eqs. (3.17)–(3.24) with $(\rho, \vec{x}, t)$ replaced by $(r, \vec{y}, \tau)$, and a_I, b_I, c_I, d_I determined from the asymptotic form (2.22). In this case the n, w, g and R independence of the resulting equations would be manifest from the beginning.

$$
\begin{aligned}
f_3(r) &= \frac{2}{r}\,\frac{1}{\sqrt{1+4\,C\,(h'(r))^2}}\,, \\
f_4(r) &= \frac{1}{\sqrt{1+4\,C\,(h'(r))^2}}\,, \\
f_5(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,C\,(h'(r))^2}\right), \\
f_6(r) &= \frac{1}{2}\,\partial_r\left(e^{h(r)}\,\sqrt{1+4\,C\,(h'(r))^2}\right). \qquad (3.56)
\end{aligned}
$$

Finally, substituting these into the third equation of (3.54) we get a differential equation for h:

$$
C\,h'\left(1+4\,C\,(h')^2\right)+C\,r\,h'' = \frac{r^2}{8}\,e^{-h}\left(1+4\,C\,(h')^2\right)^{3/2}-\frac{r}{4}\left(1+4\,C\,(h')^2\right). \qquad (3.57)
$$

The boundary condition on h follows from (2.22)

$$
e^h \simeq \frac{r}{2} \qquad \text{for large } r\,. \qquad (3.58)
$$

One can easily verify that (3.58) satisfies (3.57) for large r.

For small r eq. (3.57) admits a solution

$$
e^h \simeq \frac{r^2}{4\sqrt{C}}\,, \qquad (3.59)
$$

which leads to the solution (3.50). However since (3.57) is a second order differential equation for h, there is no *a priori* guarantee that there is a smooth solution that interpolates between (3.58) and (3.59). We shall analyze this issue in section 3.6. Note that for $C=0$ eq. (3.57) becomes a purely algebraic equation for h which admits a unique solution $e^h = r/2$ and reproduces the result of section 2.

3.6 The analysis of the interpolating solution

We shall now analyze the differential equation (3.57) for $C=1$ and analyze the possibility of a solution that interpolates between (3.58) for large r and (3.59) for small r. We shall begin by analyzing fluctuations around the asymptotic solutions. For large r, if we make the ansatz

$$
h(r) = \ln\frac{r}{2}+\phi\,, \qquad (3.60)
$$

and assume that ϕ and all its derivatives are of order unity, then (3.57) gives, for $C=1$,

$$
\phi'' = \frac{1}{4}\,e^{-\phi}\,\{1+4(\phi')^2\}^{3/2}-\frac{1}{4}\,\{1+4(\phi')^2\}\,. \qquad (3.61)
$$

$\phi=0$ is a solution of this equation as expected. For small ϕ the equation reduces to

$$
\phi'' = -\frac{1}{4}\,\phi+\mathcal{O}(\phi^2)\,, \qquad (3.62)
$$

which has, as solutions

$$
\phi = A\,\cos\left(\frac{r}{2}+B\right)+\mathcal{O}(A^2)\,, \qquad (3.63)
$$

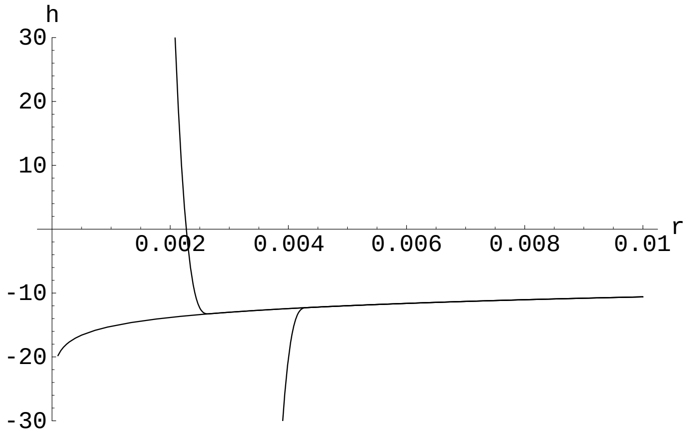

Figure 1: Trajectories neighbouring $h = 2\ln\frac{r}{2}$ for small r.

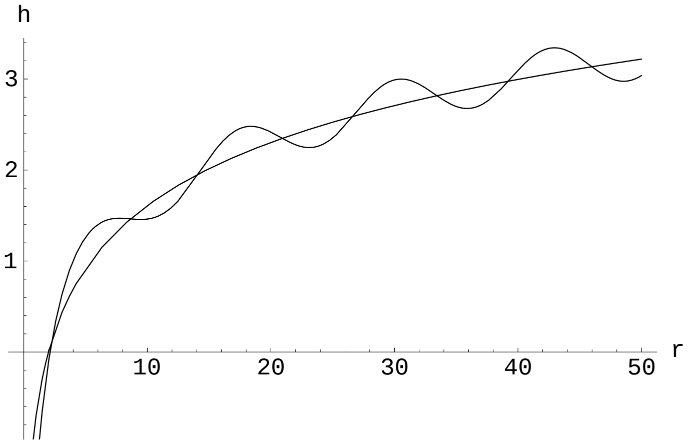

Figure 2: Numerical result for the solution to (3.57) satisfying the boundary condition $h = 2\ln\frac{r}{2}$ for small r. The smooth curve represents $h = \ln\frac{r}{2}$.

for arbitrary integration constants A, B. Thus $\phi = 0$ is an elliptic fixed point of the second order autonomous system described by (3.61) and we expect that there is a generic set of initial conditions for which the solution to (3.57) at large r will have periodic oscillations around $h = \ln\frac{r}{2}$:

$$h = \ln\frac{r}{2} + A\,\cos\left(\frac{r}{2} + B\right) + \mathcal{O}(A^2)\,. \tag{3.64}$$

Analyzing the fluctuations of the solutions around the solution (3.59) near $r = 0$ is more difficult. Numerical analysis suggests that the behaviour of the solution around $r = 0$ is highly unstable and for slight changes in the initial condition the solution develops spontaneous singularities at some value of r close to zero. This has been illustrated in figure 1 where we have displayed some trajectories neighbouring the solution (3.59) for small r.

For this reason, in order to study if there is a solution to (3.57) that interpolates between (3.58) and (3.59), we begin with the solution (3.59) for small r and numerically integrate it to study its behaviour at large r. The result is shown in figure 2. We see from this that the solution does not approach (3.58), but oscillates around it, as is expected for a generic initial condition. This seems unlikely to be a numerical error, and seems to indicate that the solution that has the correct near horizon behaviour does not approach the desired form at large r.

We shall now argue however that there is a subtlety in this interpretation and that once this subtlety is taken into account, the asymptotic behaviour of the solution is consistent with the desired form. For this we note that for small A, the solution (3.63) implies the following asymptotic forms for the f_i's:

$$\begin{aligned} f_1 \simeq \frac{r}{2}\left(1 + A\cos\left(\frac{r}{2}+B\right)\right), \quad f_2 \simeq \frac{2}{r}\left(1 - A\cos\left(\frac{r}{2}+B\right)\right), \quad f_3 \simeq \frac{2}{r}, \quad f_4 \simeq 1, \\ f_5 = f_6 \simeq \frac{1}{4} - \frac{Ar}{8}\sin\left(\frac{r}{2}+B\right) + \frac{A}{4}\cos\left(\frac{r}{2}+B\right). \end{aligned} \tag{3.65}$$

Substituting these into (2.21) we see that at large r, and to linear order in A, the modification of the solution appears only in the expression for the metric and the gauge fields. In particular, we have

$$\begin{aligned} \hat{ds}^2_{\text{string}} &\simeq -\frac{r^2}{4}\left(1 + A\cos\left(\frac{r}{2}+B\right)\right)d\tau^2 + \left(1 - A\cos\left(\frac{r}{2}+B\right)\right)d\vec{x}^2, \\ \hat{F}^{(1)}_{rt} = \hat{F}^{(2)}_{rt} &\simeq \frac{1}{4} - \frac{Ar}{8}\sin\left(\frac{r}{2}+B\right) + \frac{A}{4}\cos\left(\frac{r}{2}+B\right). \end{aligned} \tag{3.66}$$

Under the change of variable $\tau = 2\eta/r$, the metric and the gauge fields take the form

$$\begin{aligned} \hat{ds}^2_{\text{string}} &= -d\eta^2 + d\vec{x}^2 - A\cos\left(\frac{r}{2}+B\right)(d\eta^2 + d\vec{x}^2) + \mathcal{O}\left(\frac{1}{r}\right) \\ \hat{F}^{(1)}_{r\eta} &= \hat{F}^{(2)}_{r\eta} \simeq -\frac{A}{4}\sin\left(\frac{r}{2}+B\right) + \mathcal{O}\left(\frac{1}{r}\right). \end{aligned} \tag{3.67}$$

This shows that for small A the asymptotic solution differs from the (locally) flat background by an oscillatory piece proportional to A. Hence this must represent a solution of the linearized equations of motion.[14] This might seem surprising since normally the only solution of linearized equations of motion for the graviton and the gauge fields are gravitational and electromagnetic wave solutions. However in the present circumstances there can be additional solutions because the action has higher derivative terms. In order to illustrate this we consider the simpler example of a scalar field ψ with action:

$$\frac{1}{2}\int d^4x\, \psi \,\Box\left(1 - \frac{\Box}{M^2}\right)\psi\,. \tag{3.68}$$

The equations of motion for ψ has solutions of the form $A\, e^{ik.x}$ with

$$k^2 = 0 \qquad \text{or} \qquad -M^2\,. \tag{3.69}$$

[14] Although (3.56), (3.57) were derived using the requirement of supersymmetry preservation, it has been argued in [17] that a solution of these equations also satisfy the classical field equations. We have checked explicitly that the metric fluctuations given in (3.67) does satisfy the linearized equations of motion around the flat background, but we shall not demonstrate it here.

Thus a single scalar field can describe plane waves of different masses in the presence of higher derivative terms. Similar phenomenon occurs for gravity and gauge fields in the presence of higher derivative terms in the action.

Note however that the presence of such additional oscillatory solutions will, upon quantization, give rise to additional quantum states which are not present in the spectrum of string theory. Thus there is an apparent contradiction between field theory and string theory results. This problem was resolved by Zwiebach [10] who argued that these higher derivative terms should be removed by appropriate field redefinition. For example by making a field redefinition

$$g_{\mu\nu} \to g_{\mu\nu} + a\, R_{\mu\nu} + b\, R\, g_{\mu\nu}\,, \tag{3.70}$$

for appropriate constants a and b, we can ensure that the curvature squared terms appear in the action in the Gauss-Bonnet combination. This particular combination of terms has the property that when we expand this in the weak field approximation, the quadratic term involving the graviton field does not receive any contribution from the curvature squared term. As a result the linearized equations of motion of the graviton field remain unmodified and we only get the usual plane wave solutions. Since this is what string spectrum predicts, we see that this redefined metric is the correct variable to be used to make direct contact with string theory. A similar field redefinition must be carried out for the gauge fields as well.

Under such field redefinitions the oscillatory solutions of the type given in (3.67) are mapped to zero. We shall illustrate this in the context of the scalar field action (3.68). The field redefinition that brings the action to the standard action for a massless scalar field is

$$\widetilde{\psi} = \left(1 - \frac{\Box}{M^2}\right)^{1/2} \psi\,. \tag{3.71}$$

Under this map the solution $\psi = Ae^{ik.x}$ gets mapped to $Ae^{ik.x}$ for $k^2 = 0$ and to 0 for $k^2 = -M^2$. Thus in terms of the variable $\widetilde{\psi}$ only the plane wave solutions with $k^2 = 0$ are present.

This discussion shows that the fluctuations proportional to A in (3.67) are unphysical and are in fact mapped to zero when we use the correct field variables. It should be emphasized that we have not explicitly constructed the field redefinition, but are relying on the fact that the effective field theory that correctly describes tree level string theory must admit such field redefinitions. Although our discussion has been focussed at the linearized level, we expect that the result should be valid beyond the linear approximation, and that when we use the right field variables, the two parameter family of solutions of the differential equation (3.61), valid for large r, will map to a single solution. This in turn would imply that the oscillations that we see in figure 2 are due to the wrong choice of field variables, and should disappear once we make the right choice. Presumably when we use the right choice of field variables the differential equation (3.57) will be replaced by an ordinary equation with a unique solution which will interpolate correctly between the desired asymptotic limits.

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3.7 Effect of quantum corrections and holomorphic anomaly

Finally let us consider the contribution to the black hole entropy obtained after taking into account the full quantum corrections to the function $f(u)$.[15,16] In this case [11]

$$f(u) = -\frac{1}{128\,\pi\,i} \ln \Delta\left(e^{2\pi i u}\right), \tag{3.72}$$

where

$$\Delta(q) = (\eta(q))^{24}, \qquad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{3.73}$$

Eqs. (3.35), (3.40) now give

$$S_0^2 \frac{\Delta'\left(e^{-2\pi S_0}\right)}{\Delta\left(e^{-2\pi S_0}\right)} = \frac{nw}{4}, \qquad S_{BH} = 4\pi \left[S_0 \frac{\Delta'\left(e^{-2\pi S_0}\right)}{\Delta\left(e^{-2\pi S_0}\right)} - \frac{1}{2\pi} \ln \Delta\left(e^{-2\pi S_0}\right) \right], \tag{3.74}$$

where

$$\Delta'(q) \equiv q \frac{\partial \Delta(q)}{\partial q}. \tag{3.75}$$

Note that for small q

$$\ln(\Delta(q)) = \ln q + \mathcal{O}(q). \tag{3.76}$$

This gives, for large S,

$$f(iS) \simeq -\frac{i}{64} S + \mathcal{O}\left(e^{-2\pi S}\right). \tag{3.77}$$

This agrees with (3.49) for large S. Thus for large S_0 the quantum corrected answer for the entropy, computed from eqs. (3.35), (3.40), reduces to the tree level answer (3.50), (3.51), as is expected. We can take into account the corrections by solving the equations for K_0 and S_0 iteratively as a power series expansion in $e^{-2\pi S_0}$. Since to leading order S_0 is given by $\frac{1}{2}\sqrt{nw}$ we see that the quantum corrections to the black hole entropy from this special class of higher derivative terms is of order $e^{-\pi\sqrt{nw}}$ for large nw. Thus we have

$$S_{BH} = 4\,\pi\,\sqrt{nw} + \mathcal{O}\left(e^{-\pi\sqrt{nw}}\right). \tag{3.78}$$

This however is not the complete story.[17] As was pointed out in [14, 15], there are corrections to the above formula due to holomorphic anomaly [36, 37]. The effect of this is to add a non-holomorphic piece $\frac{3i}{32\pi}\ln(S+\bar{S})$ to $f(iS)$. The modified black hole entropy (see eqs.(4.12), (4.15) of [15]) reduces to, in the present case,

$$S_{BH} = 2\pi \frac{nw}{S_0 + \bar{S}_0} - 12 \ln \left[(S_0 + \bar{S}_0)\, \eta\left(e^{-2\pi S_0}\right)^4 \right], \tag{3.79}$$

[15] Note however that this takes into account only a special class of corrections and does not correspond to the full quantum corrected black hole entropy.

[16] Ref. [9] follows a somewhat different approach for relating the quantum corrected black hole entropy to the statistical entropy. This uses a mixed ensemble [32] and does not explicitly take into account the effect of holomorphic anomaly. Presumably the two approaches are related but the relationship is not completely clear to us.

[17] I would like to thank R. Gopakumar for drawing my attention to the role of holomorphic anomaly in producing logarithmic corrections.

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where now S_0 is given by the solution of the equation

$$-\frac{6}{\pi}\left[2\,\partial_{S_0}\ln\eta\left(e^{-2\pi S_0}\right)+\frac{1}{S_0+\bar{S}_0}\right]=\frac{nw}{(S_0+\bar{S}_0)^2}\,. \tag{3.80}$$

The effect of holomorphic anomaly is represented by the term proportional to $\ln(S_0+\bar{S}_0)$ in (3.79) and the term proportional to $(S_0+\bar{S}_0)^{-1}$ in (3.80). For real S_0, (3.80) gives,

$$S_0=\frac{1}{2}\sqrt{nw}+\mathcal{O}(1)\,, \tag{3.81}$$

and hence from (3.79)

$$S_{BH}=4\pi\sqrt{nw}-12\ln\sqrt{nw}+\mathcal{O}(1)\,. \tag{3.82}$$

We can try to compare this with the logarithm of the degeneracy of elementary string states. For a given n and w the degeneracy d_{nw} is determined by the formula [6, 7]:[18]

$$\frac{16}{\Delta(q)}=q^{-1}\sum_{N=0}^{\infty}d_{N-1}\,q^N\,. \tag{3.83}$$

For large-N, d_N behaves as

$$d_N\sim 8\sqrt{2}\,N^{-27/4}\,\exp(4\pi\sqrt{N})\,. \tag{3.84}$$

Thus

$$S_{\rm stat}=\ln(d_{nw})\simeq 4\,\pi\,\sqrt{nw}-\frac{27}{2}\,\ln\sqrt{nw}+\mathcal{O}(1)\,. \tag{3.85}$$

Comparing (3.78) with (3.85) we see that the quantum corrected Bekenstein-Hawking entropy does not correctly reproduce the logarithmic corrections to the statistical entropy.

We should note however that the definition of statistical entropy itself can be ambiguous when we consider non-leading corrections. Had we used the definition of statistical entropy based on a different kind of ensemble instead of the microcanonical ensemble, we would have gotten an answer that differs from (3.85). Consider for example the analog of the grand canonical ensemble where we introduce a chemical potential μ conjugate to nw and introduce the partition function

$$e^{\mathcal{F}(\mu)}=\sum_{N=0}^{\infty}d_{N-1}\,e^{-\mu(N-1)}\,. \tag{3.86}$$

Then we can define the statistical entropy through the thermodynamic relations

$$\widetilde{S}_{\rm stat}=\mathcal{F}(\mu)+\mu\,nw\,, \tag{3.87}$$

where μ is obtained by solving the equation

$$\frac{\partial\mathcal{F}}{\partial\mu}=-nw\,. \tag{3.88}$$

[18]Note the $N-1$ in the subscript of d. This is due to the fact that for given n and w, the required level of left-moving oscillators is $N=nw+1$. Thus the associated degeneracy is $d_{nw}=d_{N-1}$.

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For large nw we can approximate the sum in (3.86) as $d_{N_0-1}e^{-\mu(N_0-1)}$ where N_0 is the value of N that maximizes the summand. (3.88) now yields the result $N_0 - 1 \simeq nw$ and (3.87) gives $S_{\rm stat} \simeq \ln d_{nw}$ to leading order. This agrees with the definition of entropy based on the microcanonical ensemble. However there are non-leading corrections to this formula. To see this in the present context, note that (3.83), (3.86) give

$$\mathcal{F}(\mu) = \ln \frac{16}{\Delta(e^{-\mu})} = \ln \frac{16}{\Delta(e^{-4\pi^2/\mu}) \left(\frac{\mu}{2\pi}\right)^{-12}} = \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \ln 16 + \mathcal{O}(e^{-4\pi^2/\mu}) \,, \quad (3.89)$$

where we have used the modular transformation law of $\Delta(e^{-\mu})$ under $\mu \to 4\pi^2/\mu$. Eq. (3.88), (3.87) then give,

$$\mu = \frac{2\pi}{\sqrt{nw}} + \mathcal{O}\left(\frac{1}{\sqrt{nw}}\right) , \qquad \widetilde{S}_{\rm stat} = 4\pi \sqrt{nw} - 12 \ln \sqrt{nw} + \mathcal{O}(1) \,. \quad (3.90)$$

Thus we see that the logarithmic corrections present in (3.85) and (3.90) are different. In particular (3.90) agrees with the black hole entropy given in (3.82).

We can in fact do better and show that $\widetilde{S}_{\rm stat}$ agrees with S_{BH} up to an additive constant of $\ln 16$ and exponentially suppressed corrections. For this we note that for $\mathcal{F}(\mu)$ given in (3.89), eqs. (3.87), (3.88) take the form

$$\widetilde{S}_{\rm stat} \simeq \frac{4\pi^2}{\mu} + 12 \ln \frac{\mu}{2\pi} + \ln 16 + \mu\, nw \,, \qquad -\frac{4\pi^2}{\mu^2} + \frac{12}{\mu} \simeq -nw \,, \quad (3.91)$$

where we have only ignored corrections of order $\exp(-4\pi^2/\mu)$. On the other hand, ignoring terms of order $e^{-2\pi S_0}$, and taking S_0 to be real, eqs. (3.79), (3.80) can be rewritten as

$$S_{BH} \simeq 4\,\pi\, S_0 - 12 \ln(2S_0) + \frac{\pi nw}{S_0} \,, \qquad 4S_0^2 - \frac{12 S_0}{\pi} \simeq nw \,. \quad (3.92)$$

Comparing (3.91) and (3.92) we see that they are identical up to an additive constant of $\ln 16$ in the expression for $\widetilde{S}_{\rm stat}$, if we make the identification

$$S_0 = \frac{\pi}{\mu} \,. \quad (3.93)$$

We would like to add however that it is not *a priori* obvious which definition of the statistical entropy should be compared directly with the geometric entropy. Thus comparison of the black hole and statistical entropy beyond heterotic string tree level remains ambiguous.

4. Generalizations and open questions

1. **Other four dimensional heterotic string theories:** instead of considering toroidal compactification of heterotic string theory one could consider some other $N = 2$ or $N = 4$ compactification of heterotic string theory for which the compact manifold has the form $S^1 \times K_5$ for some compact space K_5. It was argued in [38] that in all such cases if we consider a fundamental string wrapped around the circle

S^1 carrying w units of winding and n units of momentum along S^1, the entropy of the corresponding black hole solution continues to be given by (2.25) with the same universal constant a. On the other hand the statistical entropy, computed from the spectrum of the fundamental string wrapped on S^1, is also given by the same formula (2.26) for large-N and w. Thus once the agreement between (2.25) and (2.26) has been established for the toroidally compactified heterotic string theory, it must continue to hold for all other compactifications.

2. **Higher dimensional heterotic string theories:** it was shown in [39] that the scaling argument of [8], reviewed in section 2, continues to hold for toroidally compactified heterotic string theory with higher number of non-compact dimensions. Thus it is natural to ask if the modification of the black hole solution, induced by the supersymmetric generalization of the curvature squared term in higher dimension, produces the correct coefficient of the black hole entropy so that it agrees with the statistical entropy obtained from computing the degeneracy of elementary string states. At present the answer to this question is not known.

3. **Other higher derivative corrections:** in [9, 21] as well as in the present paper we have taken into account only a specific class of higher derivative terms which arise from supersymmetrization of the curvature squared term. Since the non-trivial modification of the solution takes place at $r \sim 1$ where the higher derivative corrections are of order unity, there is no *a priori* reason why further higher derivative corrections cannot completely change the results. It will be interesing to explore if it is possible to analyze the effect of these higher derivative terms by directly working with the σ-model that describes string propagation in this background [40].

4. **Type-II string theories:** one can carry out a similar analysis for toroidal or other compactification of type-II superstring theories which have at least two supersymmetries from the right-moving sector of the world-sheet and for which the compact space has a free circle on which one can wrap the fundamental string. It was shown in [38] that the scaling argument of [8] can be generalized to this case to yield a formula similar to (2.25):
$$S_{BH} = a' \sqrt{nw} \,, \tag{4.1}$$
where the constant a' is universal for all superstring compactifications of the type mentioned above, but could differ from the constant a for heterotic string compactifications. On the other hand the calculation of the degeneracy of the elementary string states yields the following expression for the statistical entropy:
$$S_{\rm stat} = 2\sqrt{2}\,\pi\sqrt{nw} \,. \tag{4.2}$$
Thus it is natural to ask if higher derivative corrections similar to the one studied here could give rise to the $a' = 2\sqrt{2}$ relation in superstring theory.

Unfortunately however tree level type-II string theory has no curvature squared term of the type discussed here and as a result the analog of the term that gave the correct value of a in heterotic string compactification does not exist in type-II string theory. Hence a' continues to vanish. The resolution of this puzzle is not clear to us. It is of course possible that type-II string theory will have other higher derivative corrections which modify the solution and gives us a finite entropy in agreement with (4.2), but then the question that would arise is: why are such corrections not present in the heterotic string theory?

Although we do not have an answer to this puzzle, the following observation may be useful. First note that the geometric entropy formula given in [13, 17] always agrees with the apparent statistical entropy computed in [31]. Thus a discrepancy between the geometric entropy and statistical entropy can be regarded as a mismatch between the apparent statistical entropy computed in [31] and the correct statistical entropy, and understanding the origin of the latter disagreement may give us some insight into the origin of the former discrepancy. To this end we note that the analysis of [31] was carried out by describing the theory under consideration as an M-theory compactified on $S^1 \times K_6$ for some six dimensional Calabi-Yau manifold K_6, and the system whose entropy is being computed as an M5-brane wrapped on $S^1 \times K_4$ where K_4 is a 4-cycle in K_6. One then takes the limit of large S^1 to regard this as a string wrapped on S^1, where the string is identified as the M5-brane wrapped on K_4. The degeneracy of BPS states, with the string carrying certain momentum along S^1, is then given by $\exp(2\pi\sqrt{c_L n/6})$ where c_L is the central charge associated with the left moving modes on the string. Thus the statistical entropy is given by $2\pi\sqrt{c_L n/6}$.

The subtlety in this computation lies in the determination of c_L. This requires knowing the number of left-moving massless modes living on the M5-brane wrapped on K_4. In [31] this computation was done by using certain genericity assumption under which the computation of the number of massless degrees of freedom reduces to computation of certain topological index. However in a non-generic case the number of massless modes may differ from this index, and in that case the entropy formula given in [31] will not be correct. Since the entropy formula of [31] is identical to the formula for the geometric entropy computed in [13, 17], this would imply that in these cases the geometric entropy formula of [13, 17] will differ from the statistical entropy.

Let us now examine the computation of [31] both for the case of heterotic string on $T^4 \times \widetilde{S}^1 \times S^1$ and type IIA on $T^4 \times \widetilde{S}^1 \times S^1$. In the first case using string-string duality [41 – 45] we can map the theory to type IIA on $K3 \times \widetilde{S}^1 \times S^1$, which in turn is equivalent to M-theory on $\widehat{S}^1 \times K3 \times \widetilde{S}^1 \times S^1$. Under this duality the fundamental heterotic string wrapped on S^1 gets mapped to M5-brane wrapped on $K3 \times S^1$. In this case the formula given in [31] gives $c_L = c_2(K3)$ where $c_2(\mathcal{M})$ denotes the second Chern class of $\mathcal{M}$. Since $c_2(K3) = 24$, we get the correct answer for the central charge associated with the left-moving degrees of freedom of a fundamental heterotic string. As a result geometric entropy agrees with the statistical entropy.

On the other hand using an analog of the string-string duality formula we can map type IIA on $T^4 \times \widetilde{S}^1 \times S^1$ to type IIA on $\widetilde{T}^4 \times \widetilde{S}^1 \times S^1$ or M-theory on $\widehat{S}^1 \times \widetilde{T}^4 \times \widetilde{S}^1 \times S^1$ so that the fundamental type-IIA string wrapped on S^1 in the first theory gets mapped to the M5-brane wrapped on $\widetilde{T}^4 \times S^1$ [46]. The formula given in [31] now gives $c_L = c_2(T^4) = 0$. This clearly is not the correct answer for the central charge of the left-moving modes on a type-IIA string. This is responsible for the disagreement between the geometric entropy formula of [13, 17] and the statistical entropy.

This analysis shows that agreement between the geometric entropy computed in [13, 17] and the statistical entropy depends on whether the genericity assumption of [31] holds or not. Thus in order to argue that the geometric entropy always reproduces the statistical entropy, we need to show that when the genericity assumption holds, there is a non-renormalization theorem that prevents any correction to the entropy formula by higher derivative terms which were not included in the analysis of [13, 17]. On the other hand, when the genericity assumption fails, the non-renormalization theorems must also break down, and the higher derivative terms should become important.

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